Divisor graphs have arbitrary order and size

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Abstract

A divisor graph G is an ordered pair (V, E) where $V \subset \mathbb{Z}$ and for all $u \neq v \in V$, $uv \in E$ if and only if $u \mid v$ or $v \mid u$. A graph which is isomorphic to a divisor graph is also called a divisor graph. In this note, we will prove that for any $n \geqslant 1$ and $0 \leqslant m \leqslant \binom{n}{2}$ then there exists a divisor graph of order n and size m. We also present a simple proof of the characterization of divisor graphs which is due to Chartran, Muntean, Saenpholpant and Zhang.

1 Introduction

The notion of divisor graph was first introduced by Singh and Santhosh [2]. A divisor graph $G(V)$ is an ordered pair (V, E) where $V \subset \mathbb{Z}$ and for all $u \neq v \in V$, $uv \in E$ if and only if $u \mid v$ or $v \mid u$. A graph which is isomorphic to a divisor graph is also called a divisor graph. The main result of this note is the following theorem.

Theorem 1 *For any $n \geqslant 1$ and $0 \leqslant m \leqslant \binom{n}{2}$ then there exists a divisor graph of order n and size m.*

To prove Theorem 1, we need the following characterization of divisor graphs is due to Chartran, Muntean, Saenpholpant and Zhang [1].

Theorem 2 *A graph G is divisor graph if and only if there is an orientation D of G such that if $(x, y), (y, z)$ are edges of D then so is (x, z).*

The proof of Theorem 2 in [1] is by induction on the order of the graph. For the completeness of this note, we will present a simple (and direct) proof of this theorem in Section 3. From Theorem 2, we introduce the definition of a divisor digraph which will be usefull in the proof of Theorem 1.

Definition 1 *A digraph G is a divisor digraph if and only if $(x, y), (y, z)$ are edges of G then so is (x, z).*

It is clear that if G is a divisor digraph then the graph obtained by ignoring the direction of edges of G is a divisor graph.

2 Proof of Theorem 1

Suppose that $G = (V, E)$ is a graph with vertex set $V = \{v_1, \ldots, v_n\}$ and size m. The degree of a vertex v_i is the number of edges of G incident with v_i. Let $d_1 \geqslant d_2 \geqslant \ldots \geqslant d_n$ be the vertex degrees in non-increasing order and let $f_G = |\{i : d_i \geqslant i\}|$. Let e_i be the number of vertices with degree at least i, that is $e_i = |\{j : d_j \geqslant i\}|$. Then we have $e_1 \geqslant e_2 \geqslant \ldots \geqslant e_n$. Moreover, we also have

$$d_i = |\{j : e_j \geqslant i\}|. \tag{1}$$

We have the following lemmas.

Lemma 1 *Let $n - 1 = d_1 \geqslant \ldots \geqslant d_n \geqslant 1$ be a sequence of natural numbers and let $e_i = |\{j : d_j \geqslant i\}|$. Suppose that*

$$\sum_{i=1}^{n} d_i = 2m, \tag{2}$$

and

$$\sum_{i=1}^{t} d_i = \sum_{i=1}^{t} (e_i - 1) \tag{3}$$

for all $1 \leqslant t \leqslant f = |\{i : d_i \geqslant i\}|$. Then there exists a divisor graph of order n and size m.

Proof We construct a digraph G with vertex set $v_1, \ldots, v_n$ as follows. For $1 \leqslant i \leqslant f$, then (v_i, v_j) is an edge of G for $i + 1 \leqslant j \leqslant d_i + 1$. We first show that G is a graph of size m. It suffices to show that $\deg(v_i) = d_i$ for all i (where $deg(v_i)$ is the number of incident edges of vertex v_i, regardless of their directions). We have three cases.

1. Suppose that $1 \leqslant i \leqslant f$. Then it is clear from the construction that $\deg(v_i) = d_i$.

2. Suppose that $i = f + 1$. Then $d_i = d_{f+1} \leqslant f$ (by the definition of f). From (3), we have $d_j = e_j - 1$ for $1 \leqslant j \leqslant f$. If $d_{f+1} < f$ then $e_f \leqslant f$, or

$$f \leqslant d_f = e_f - 1 < f,$$

which is a contradiction. Hence $d_{f+1} = f$. For $1 \leqslant j \leqslant f$ then $d_j + 1 \geqslant d_f + 1 \geqslant f + 1 = i$. So (v_j, v_i) is an edge of G and $\deg(v_i) = f$. Thus, $\deg(v_i) = d_i$.

3. Suppose that $i > f + 1$. Then (v_j, v_i) is an edge of G if and only if $1 \leqslant j \leqslant f$ and $j + 1 \leqslant i \leqslant d_j + 1$. For $j > f$ we have $d_j \leqslant d_{f+1} < f + 1 \leqslant i$. This implies that

$$\deg(v_i) = |\{1 \leqslant j \leqslant d : d_j \geqslant i - 1\}| = |\{j : d_j \geqslant i - 1\}| = e_{i-1}$$

for all $i > f + 1$. From (1), we have $d_i = e_i - 1$ for $1 \leqslant i \leqslant f$. For $i > f + 1$ then $d_j \geqslant i - 1$ or $e_j \geqslant i$ only if $j \leqslant f$ (since $d_{f+1} = f < i - 1$). Hence

$$\begin{aligned} e_{i-1} &= |\{j : d_j \geqslant i - 1\}| \\ &= |\{1 \leqslant j \leqslant f : d_j \geqslant i - 1\}| \\ &= |\{1 \leqslant j \leqslant f : e_j - 1 \geqslant i - 1\}| \\ &= |\{j : e_j \geqslant i\}| = d_i. \end{aligned}$$

Thus, $\deg(v_i) = e_{i-1} = d_i$ for all $i > f + 1$.

Therefore, we have $\deg(v_i) = d_i$ for $1 \geqslant i \geqslant n$. This implies that G has order n and size m.

Now, we will show that G is a divisor digraph. Suppose that (v_i, v_j) and (v_j, v_k) are two edges of G. Then from the above construction, $1 \leqslant i, j \leqslant d$ and $k \leqslant d_j + 1 \leqslant d_i + 1$. Thus (v_i, v_k) is also an edge of G. This implies that G is a divisor digraph. Let H be the graph obtained from G by ignoring the direction of edges of G. Then H is a divisor graph of order n and size m. This concludes the proof of the lemma. $\square$

Lemma 2 *Let $n - 1 = d_1 \geqslant \ldots \geqslant d_n \geqslant 1$ be a sequence of natural numbers and set $e_i = |\{j : d_j \geqslant i\}|$. Suppose that*

$$\sum_{i=1}^{n} d_i = 2m < n(n-1), \tag{4}$$

and

$$\sum_{i=1}^{t} d_i = \sum_{i=1}^{t} (e_i - 1) \tag{5}$$

for all $1 \leqslant t \leqslant f = |\{i : d_i \geqslant i\}|$. Then there exists a sequence $n - 1 = d'_1 \geqslant \ldots \geqslant d'_n \geqslant 1$ of natural numbers such that

$$\sum_{i=1}^{n} d'_i = 2(m+1), \tag{6}$$

and

$$\sum_{i=1}^{t} d'_i = \sum_{i=1}^{t} (e'_i - 1) \tag{7}$$

for all $1 \leqslant t \leqslant f' = |\{i : d'_i \geqslant i\}|$, where $e'_i = |\{j : d_j \geqslant i\}|$ for $1 \leqslant i \leqslant n$.

Proof If $f \geqslant n - 1$ then $n - 1 \geqslant d_1 \geqslant \ldots \geqslant d_{n-1} \geqslant d_f \geqslant f = n - 1$. Hence $d_1 = \ldots. = d_{n-1} = n - 1$, and

$$\sum_{i=1}^{n-1} e_i = n(n-1).$$

We have $e_i \leqslant n$ for $1 \leqslant i \leqslant n - 1$, so $e_1 = \ldots = e_{n-1} = n$. Hence $d_i \geqslant n - 1$ for $1 \leqslant i \leqslant n$ or

$$\sum_{i=1}^{n} d_i = n(n-1),$$

which is a contradiction. Thus, $f < n-1$. Let g be the smallest index such that $d_i < n-1$. Then we have $2 \leqslant g \leqslant f + 1$ (since $d_{f+1} \leqslant f < n - 1$). We have two cases.

1. Suppose that $2 \leqslant g \leqslant f$. Set $h = d_g + 2$, $d'_g = d_g + 1$, $d'_h = d_h + 1$ and $d'_i = d_i$ for $i \neq g, h$. We have (6) holds since

$$\sum_{i=1}^{n} d'_i = 2 + \sum_{i=1}^{n} d_i = 2(m+1).$$

Recall that from the proof of Lemma 1, we have

$$d_i = \begin{cases} e_i - 1 & \text{if } i \leqslant f \\ f & \text{if } i = f+1 \\ e_{i-1} & \text{if } i > f+1. \end{cases} \tag{8}$$

Since $d_g = h-2 \geqslant g$ and $g \leqslant f$, we have $e_g = h-1$. This implies that $d_{h-1} \geqslant g$ and $d_h < g$. Besides, $d_1 = \ldots = d_{g-1} = n-1$ so $d_h \geqslant d_n = e_{n-1} \geqslant g-1$. Hence $d_h = g-1$, $d'_g = h-1$ and $d'_h = g$. Therefore, we have $e'_g = e_g + 1$, $e'_h = e_h + 1$ and $e'_i = e_i$ for $i \neq g, h$. We have $d_h = g-1 \leqslant f-1$ so $h > f+1$. Hence $d'_{f+1} = d_{f+1} < f+1$ or $f' \leqslant f$. And we have (7) holds for $1 \leqslant t \leqslant f'$ since (5) holds for $1 \leqslant t \leqslant f' \leqslant f$.

2. Suppose that $g = f+1$. Then from (8) we have $d_g = f = g-1$. Set $h = f+2$, $d'_g = d_g + 1$, $d'_h = d_h + 1$ and $d'_i = d_i$ for $i \neq g, h$. Then it is clear that (6) holds. Besides, we have $d_1 = \ldots = d_{g-1} = n-1$ so

$$g - 1 = d_g \geqslant \ldots \geqslant d_n = e_{n-1} \geqslant g-1.$$

Hence $d_g = \ldots = d_n = g-1$. We have $d'_g = d'_h = g < h$, so $f' = f+1$, $e'_g = e_g + 2 = g+1$ and $e'_i = e_i$ for $i \neq g$. From (5), we have

$$\sum_{i=1}^{t} d'_i = \sum_{i=1}^{t} (e'_i - 1)$$

for $1 \leqslant t \leqslant f$. We only need to check for $t = f'(= f+1 = g)$. We have

$$\sum_{i=1}^{g} d'_i = g + \sum_{i=1}^{f} d'_i = e'_g - 1 + \sum_{i=1}^{f} (e'_i - 1) = \sum_{i=1}^{g} (e'_i - 1).$$

Thus, (7) holds for $1 \leqslant t \leqslant f'$.

This concludes the proof of the lemma. □

We are now ready to prove Theorem 1. From Lemma 1, we start with the sequence $(n-1, 1, \ldots, 1)$ to obtain a divisor graph of size $n-1$. Then apply Lemma 1 and Lemma 2 inductively to obtain divisor graphs of order $n, \ldots, \binom{n}{2}$. To construct a divisor graph of order n and size m with $m < n-1$, we choose a vertex and join it with m other vertices. Thus, there exists a divisor graph of order n and size m for any n and $0 \leqslant m \leqslant \binom{n}{2}$. This concludes the proof of the theorem.

Remark 1 An interesting and open question is to find necessary and sufficient conditions for a non-increasing sequence $n-1 \geqslant d_1 \geqslant \ldots \geqslant d_n \geqslant 1$ such that there exists a divisor graphs with degree sequence $(d_1, \ldots, d_n)$.

3 Proof of Theorem 2

Suppose that G is a divisor graph. Then there exists a set V of positive integer such that $G \simeq G(V)$. We give an orientation on each edge (i, j) of G as follows

$$(i,j) \in E(G),\ i \to j \quad \text{if and only if} \quad i \mid j.$$

Suppose that $(x, y), (y, z)$ are edges of D. Then $x \mid y$ and $y \mid z$. Hence $x \mid z$ and (x, z) is an edge of G.

Now suppose that there exists an orientation D of G such that if $(x, y), (y, z)$ are edges of D then so is (x, z). We will show that G is a divisor graph. We will give an explicit labelling for G. We start with any vertex of G and label it by $\{a_1\}$ (a list of one symbol). Suppose that we have labelled k vertices of G and we have used $a_1, \ldots, a_l$ symbols (each vertex is labelled by a list of symbols and we will update this list in each step). We choose any unlabelled vertex, says v. Consider two sets

$$D_I(v) = \{u \in V(G) \mid (u, v) \in E(D)\},$$
$$D_O(v) = \{u \in V(G) \mid (v, u) \in E(D)\}.$$

We label v by $L_v = \{a_{l+1}\}$. For each $u \in D_I(v)$ and u was labelled by a list L_u then we add this list into the list L_v to have a new list L_v for v. And for each $u \in D_O(v)$ which was labelled by a list L_u then we add the new list L_v into L_u to have a new list for u. For each updated vertex u, we consider the set $D_O(u)$. For each $w \in D_O(u)$ which was labelled by a list L_w, we add the new list L_u into the list L_w to have a new list for w. We keep doing until we have no vertex to update or we come back to some vertex which we met along the way. But in the latter case, we have a sequence of vertices, says $w_1, \ldots, w_t$ such that $w_1 \to w_2 \to \ldots \to w_t \to w_1$ in D. This implies that

$$w_1 \mid w_2, w_2 \mid w_3, \ldots, w_t \mid w_1.$$

Hence $w_1 = \ldots = w_t$, which is a contradiction. Thus the process must be stopped. We repeat the process until all the vertices of G have been labelled by lists of symbols. Suppose that we have used r symbols $a_1, \ldots, a_r$. We choose r distinct primes $p_1, \ldots, p_r$ and for each vertex $v \in V(G)$ which is labelled by a list $\{a_{i_1}, \ldots, a_{i_v}\} \subseteq \{a_1, \ldots, a_r\}$ then we label it by the number

$$n(v) = p_{i_1} \ldots p_{i_v}.$$

From the construction above, if (u, v) is an edge of G then either $L_u \subset L_v$ or $L_v \subset L_u$. This implies that either $u \mid v$ or $v \mid u$. Hence G is a divisor graph. This concludes the proof of the theorem.

4 Acknowlegement

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References

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